Probabilistic methods applied to Geotechnical Engineering
1. Research motivation

2. Overview of the probabilistic analysis

3. The Point Estimate Method (PEM)

4. PEM application to geotechnical problems

5. Conclusions
1. Research motivation

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100% Uncertainty
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Geotechnical uncertainties

- Geological anomalies
- Inherent spatial variability of soil properties
- Scarcity of representative data
- Changing environmental conditions
- Unexpected failure mechanisms
- Simplifications and approximations adopted in geotechnical models
- Human mistakes in design and construction

Deterministic analysis leads to extremely conservative design with significant failure probability:

\[ \downarrow \]

unable to account for uncertainties in material and load properties.
How to deal with uncertainties?

“Uncertainty is inevitable”

Lack of perfect knowledge concerning phenomena and processes involved in problem definition and resolution.

Implementation of probabilistic analysis required

- uncertainties rationally quantified and systematically incorporated into the design process,

- means to evaluate uncertainties influence on the likelihood of **satisfactory performance** for an engineering system.

La Conchita, California
1. Engineers’ training in probability theory often limited to basic information during their early years of education.

2. Less comfortable dealing with probabilities than with deterministic analysis.

3. Common misconception that it requires significantly more data, time and effort.

4. Few published studies illustrate its implementation and benefits.

Deterministic analysis and probabilistic approach as complementary measures of acceptable design !!!

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2. Overview of the probabilistic analyses

Choice of geotechnical problems

Application of probabilistic methods

Results in terms of statistics values and probability distribution function

Comparison with Monte Carlo Method
Probabilistic methods analysed

- The First Order Second Moment Method (FOSM)
- The Second Order Second Moment Method (SOSM)
- The Point Estimate Method (PEM)
- Monte Carlo Simulations (MC)
3. The Point Estimate Method
(Rosenblueth, 1975)

Computationally straightforward technique for uncertainty analysis:
capable of estimating statistical values of a model output involving several stochastic variables,
correlated or uncorrelated, symmetric or non-symmetric.

Weighted average method similar to numerical integration formulas involving “sampling points”
and “weighting parameters”.

- Requires little knowledge of probability concepts and applies for any probabilistic distribution.
- Widely applied for reliability analysis and evaluation of failure probability.

Aim: replace probability distributions for continuous random variables with discrete equivalent
functions having the same mean value, standard deviation and skewness coefficient!
Procedure for implementing the PEM

1. Consider a relationship between performance function \( f(X_i) \) and input random variables.

2. Compute locations of sampling points \( (2^n \text{ calculations}) \):

\[
\xi_{X_i,+} = \frac{v_{X_i}}{2} + \left( 1 + \left( \frac{v_{X_i}}{2} \right)^2 \right)^{\frac{1}{2}}
\]

\[
\xi_{X_i,-} = \xi_{X_i,+} - v_{X_i}
\]

\[
x_i- = \mu_{X_i} - \xi_{X_i,-} \cdot \sigma_{X_i}
\]

\[
x_i+ = \mu_{X_i} + \xi_{X_i,+} \cdot \sigma_{X_i}
\]
Procedure for implementing the PEM

1. Consider a relationship between performance function \( f(X_i) \) and input random variables.

2. Compute locations of sampling points \( (2^n \) calculations) :

   \[
   \xi_{X_i+} = \frac{\nu_{X_i}}{2} + \left( 1 + \left( \frac{\nu_{X_i}}{2} \right)^2 \right)^{\frac{1}{2}} \\
   \xi_{X_i-} = \xi_{X_i+} - \nu_{X_i} \\
   x_{i-} = \mu_{X_i} - \xi_{X_i-} \cdot \sigma_{X_i} \\
   x_{i+} = \mu_{X_i} + \xi_{X_i+} \cdot \sigma_{X_i}
   \]

3. Determine the weights \( P_i \) (probability concentrations) to obtain all the point estimates.

   For a single random variable:

   \[
   P_{X_i+} = \frac{\xi_{X_i-}}{\xi_{X_i+} + \xi_{X_i-}} \\
   P_{X_i-} = 1 - P_{X_i+}
   \]

   Associated weights:

   \[
   P_{X_{i+2}} = P_{X_{i1}} \cdot P_{X_{i2}}
   \]
Procedure for implementing the PEM

4. Determine the performance function value \( f(X_i) \) at each sampling point locations.

<table>
<thead>
<tr>
<th>Sign</th>
<th>( P_i )</th>
<th>( c' )</th>
<th>( \tan(\phi') )</th>
<th>( q' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>++</td>
<td>0.043</td>
<td>14.647</td>
<td>0.523</td>
<td>653.548</td>
</tr>
<tr>
<td>+ -</td>
<td>0.457</td>
<td>3.003</td>
<td>0.409</td>
<td>194.171</td>
</tr>
<tr>
<td>- +</td>
<td>0.043</td>
<td>14.647</td>
<td>0.409</td>
<td>391.703</td>
</tr>
<tr>
<td>- -</td>
<td>0.457</td>
<td>3.003</td>
<td>0.523</td>
<td>365.685</td>
</tr>
</tbody>
</table>

5. Determine the first three moments of the performance function:

\[
\mu_{f(X_i)} = \sum_{i=1}^{2^n} P_i \cdot f(X_i)
\]

\[
\sigma^2_{f(X_i)} = \sum_{i=1}^{2^n} P_i \cdot (f(X_i) - \mu_{f(X_i)})^2
\]

\[
v_{f(X_i)} = \frac{1}{\sigma^3_{f(X_i)}} \sum_{i=1}^{2^n} P_i \cdot (f(X_i) - \mu_{f(X_i)})^3
\]
4. PEM application to geotechnical problems

**Terzaghi’s Bearing Capacity:** shallow foundation on a cohesive homogeneous soil

- PEM with correlated input random variables.
- PEM with uncorrelated input random variables.
- Results comparison.

**Slope Stability Analysis:** fill embankment on undrained clay

- PEM with uncorrelated input random variables.
- Results comparison.
Terzaghi’s bearing capacity

\[ q_f = c' \cdot N_c + q \cdot N_q + \frac{1}{2} \cdot \gamma \cdot B \cdot N_\gamma \]

**Gaussian (normal) distribution**

- \( \mu_{\tan \phi'} = 0.47 \)
- \( \sigma_{\tan \phi'} = 0.06 \)
- \( \nu_{\tan \phi'} = 0 \)

**Lognormal distribution**

- \( \mu_c = 4 \text{ kPa} \)
- \( \sigma_c = 3.3 \text{ kPa} \)
- \( \nu_c = 3.1 \)

**Example Calculation**

- \( B = 2 \text{ m} \)
- \( q_f \)
- \( q = 10 \text{ kPa} \)
- \( \gamma = 15 \text{ kN/m}^3 \)

- \( q \) applied over a width of 2 m

- \( \gamma \) is the unit weight of the soil

- \( c' \) is the cohesion

- \( \phi' \) is the effective friction angle

- \( N_c, N_q, N_\gamma \) are factors accounting for the uncertainties in the cohesion, skin friction, and unit weight, respectively.
PEM results with **uncorrelated** ($\rho_{\tan \phi'} = 0$) input variables

<table>
<thead>
<tr>
<th>$\mu_{q_f}$ (kPa)</th>
<th>$\sigma_{q_f}$ (kPa)</th>
<th>$\nu_{q_f}$</th>
<th>$\text{COV}_{q_f}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>305.39</td>
<td>110.48</td>
<td>1.2</td>
<td>0.36</td>
</tr>
</tbody>
</table>

![Graph showing the probability density function of $q_f$.](image)

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PEM results with correlated input variables

Rosenblueth (1981) – Two random variables

\[
P_{s_1s_2} = P_{X_1} \cdot P_{X_2} + s_1 \cdot s_2 \cdot \left( \rho_{X_1X_2} \left( \left( 1 + \left( \frac{\nu_{X_1}}{2} \right) \right) \left( 1 + \left( \frac{\nu_{X_2}}{2} \right) \right) \right) \right)^{1/2}
\]

<table>
<thead>
<tr>
<th>(\rho (\tan \phi', c'))</th>
<th>Mean value (q)</th>
<th>Standard deviation (q)</th>
<th>Skewness (q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0</td>
<td>294.110</td>
<td>34.755</td>
<td>0.000</td>
</tr>
<tr>
<td>-0.9</td>
<td>295.279</td>
<td>46.193</td>
<td>0.614</td>
</tr>
<tr>
<td>-0.8</td>
<td>296.449</td>
<td>55.290</td>
<td>0.678</td>
</tr>
<tr>
<td>-0.7</td>
<td>297.618</td>
<td>63.067</td>
<td>0.647</td>
</tr>
<tr>
<td>(\boxed{-0.6})</td>
<td>298.788</td>
<td>69.966</td>
<td>0.594</td>
</tr>
<tr>
<td>-0.5</td>
<td>299.957</td>
<td>76.225</td>
<td>0.537</td>
</tr>
<tr>
<td>-0.4</td>
<td>301.127</td>
<td>81.991</td>
<td>0.483</td>
</tr>
<tr>
<td>-0.3</td>
<td>302.297</td>
<td>87.361</td>
<td>0.432</td>
</tr>
<tr>
<td>-0.2</td>
<td>303.466</td>
<td>92.406</td>
<td>0.385</td>
</tr>
<tr>
<td>-0.1</td>
<td>304.636</td>
<td>97.174</td>
<td>0.341</td>
</tr>
<tr>
<td>0.0</td>
<td>305.805</td>
<td>101.706</td>
<td>0.300</td>
</tr>
<tr>
<td>0.1</td>
<td>306.975</td>
<td>106.032</td>
<td>0.262</td>
</tr>
<tr>
<td>0.2</td>
<td>308.144</td>
<td>110.175</td>
<td>0.226</td>
</tr>
<tr>
<td>0.3</td>
<td>309.314</td>
<td>114.157</td>
<td>0.193</td>
</tr>
<tr>
<td>0.4</td>
<td>310.484</td>
<td>117.992</td>
<td>0.161</td>
</tr>
<tr>
<td>0.5</td>
<td>311.653</td>
<td>121.695</td>
<td>0.131</td>
</tr>
<tr>
<td>0.6</td>
<td>312.823</td>
<td>125.278</td>
<td>0.102</td>
</tr>
<tr>
<td>0.7</td>
<td>313.992</td>
<td>128.751</td>
<td>0.075</td>
</tr>
<tr>
<td>0.8</td>
<td>315.162</td>
<td>132.122</td>
<td>0.049</td>
</tr>
<tr>
<td>0.9</td>
<td>316.331</td>
<td>135.399</td>
<td>0.024</td>
</tr>
<tr>
<td>1.0</td>
<td>317.501</td>
<td>138.589</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Influence of the correlation coefficient on PEM results of the bearing capacity

If $\rho_{\tan \phi' c'}$ increases then p.d.f. is wider and probability values slightly decrease.

If $\rho_{\tan \phi' c'}$ decreases then p.d.f. is narrower and probability values increase.
Comparison of PEM and MC results

<table>
<thead>
<tr>
<th>Method</th>
<th>$\mu_{q_i}$ (kPa)</th>
<th>$\sigma_{q_i}$ (kPa)</th>
<th>$\nu_{q_i}$</th>
<th>COV$_{qf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo</td>
<td>304.72</td>
<td>115.94</td>
<td>1.9</td>
<td>0.38</td>
</tr>
<tr>
<td>PEM $\rho = 0$</td>
<td>305.39</td>
<td>110.48</td>
<td>1.2</td>
<td>0.36</td>
</tr>
<tr>
<td>PEM $\rho = -0.6$</td>
<td>298.79</td>
<td>69.97</td>
<td>0.59</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Probability density function

- PEM $\rho = 0$
- PEM $\rho = -0.6$
- Monte Carlo $\rho = 0$
- FS = 2
- Deterministic mean value: 287.6 kPa

Bearing capacity $q_f$ (kPa)
Slope stability analysis

Bishop’s simplified method of slices

\[
FS = \frac{\sum_{i=1}^{n} [c_i \cdot b_i + W_i \cdot \tan \varphi_i] \cdot \frac{1}{m_{\alpha(i)}}}{\sum_{i=1}^{n} W_i \cdot \sin \alpha_i}
\]

Input uncorrelated soil parameters

<table>
<thead>
<tr>
<th>Layer</th>
<th>Mean Value</th>
<th>Standard deviation</th>
<th>Coefficient of Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Fill</strong></td>
<td>$\gamma = 20 \text{ kN/m}^3$</td>
<td>1</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>$\phi = 30^\circ$</td>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td><strong>Crust</strong></td>
<td>$\gamma = 18.81 \text{ kN/m}^3$</td>
<td>0.94</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>$c_u = 40 \text{ kPa}$</td>
<td>10</td>
<td>0.25</td>
</tr>
<tr>
<td><strong>Marine Clay</strong></td>
<td>$\gamma = 18.81 \text{ kN/m}^3$</td>
<td>0.94</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>$c_u = 34.5 \text{ kPa}$</td>
<td>6.89</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Lacustrine Clay</strong></td>
<td>$\gamma = 20.31 \text{ kN/m}^3$</td>
<td>0.99</td>
<td>0.049</td>
</tr>
<tr>
<td></td>
<td>$c_u = 31.2 \text{ kPa}$</td>
<td>9.98</td>
<td>0.32</td>
</tr>
</tbody>
</table>

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Comparison of PEM and MC results

<table>
<thead>
<tr>
<th>Method applied</th>
<th>$\mu_{FS}$</th>
<th>$\sigma_{FS}$</th>
<th>COV$_{FS}$</th>
<th>$\nu_{FS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo</td>
<td>1.462</td>
<td>0.279</td>
<td>0.191</td>
<td>0.012</td>
</tr>
<tr>
<td>PEM</td>
<td>1.535</td>
<td>0.286</td>
<td>0.186</td>
<td>0.00004</td>
</tr>
</tbody>
</table>

The diagram shows a comparison of Monte Carlo and PEM results for factor of safety. The table lists the following parameters:

- $\mu_{FS}$: Deterministic mean value
- $\sigma_{FS}$: Standard deviation
- COV$_{FS}$: Coefficient of variation
- $\nu_{FS}$: Skewness

The Monte Carlo method produced a factor of safety of 1.462 with a standard deviation of 0.279, whereas the PEM (Rosenblueth, 1975) method resulted in a factor of safety of 1.535 with a standard deviation of 0.286. The two methods show slight differences in their outcomes, with the PEM method slightly overestimating the factor of safety compared to Monte Carlo.
5. Conclusions

PEM advantages vs. MC simulations

Reasonably robust and satisfactorily accurate for a wide range of practical problems!

- Results as reliable and accurate as MC simulations.
- Smaller computational effort for a comparable degree of accuracy.
- No need of knowledge of p.d.f. shape of input random variables.
- Behaviour of non-linear function well captured.
5. Conclusions

How to cope with PEM drawbacks

➢ Performance function p.d.f. to be assumed, thus introducing uncertainty.

*In Soil Mechanics normal and lognormal distributions frequently result as output of probabilistic analysis.*

➢ If more accuracy required than larger number of input variables necessary, i.e. number of required evaluations too high to be implemented practically.

*Rosenblueth approximation method (1981) for Gaussian distributed uncorrelated input variables.*

➢ Results poor and not accurate for discontinuous functions or functions having discontinuous first derivatives and for large COVs of input variables.

*Typical geotechnical problems described by continuous functions, whose “non-linearity” not difficult to be treated by PEM. Small COV values in geotechnical literature, frequently lower than the unity.*
5. Conclusions

Final observations

- Different probabilistic methods applied to geotechnical problems to find out the most suitable one for the geotechnical field and for a further analysis.

- PEM gave as reliable and accurate results as MC with less computational effort.

- It was easily applied to a multivariate problem.

- Output as a lower bound for the evaluation of failure probability, because effects of other factors not included in the analysis.

- Spatial variability ignored to simplify the calculations assuming perfect autocorrelation, obtaining more conservative results.

**PEM: simple, but powerful technique for uncertainty analysis.**

*Its use in geotechnical reliability analysis justified by experience and theory.*
Thanks for your attention!

For further clarifications and discussion, please contact

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